

Thin buildings

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Let X be a building of uniform thickness $q + 1$. L^2 –Betti numbers of X are reinterpreted as von-Neumann dimensions of weighted L^2 –cohomology of the underlying Coxeter group. The dimension is measured with the help of the Hecke algebra. The weight depends on the thickness q . The weighted cohomology makes sense for all real positive values of q , and is computed for small q . If the Davis complex of the Coxeter group is a manifold, a version of Poincaré duality allows to deduce that the L^2 –cohomology of a building with large thickness is concentrated in the top dimension.

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Introduction

Let (G, B, N, S) be a BN –pair, and let X be the associated building (notation as in Brown [2, Chapter 5]). There are many geometric realizations of X . We consider the one introduced by Davis in [4]. Then X is a locally finite simplicial complex, acted upon by G . The action has a fundamental domain with stabiliser B . The standard choice of such a domain is called the Davis chamber. We can and will assume that G is a closed subgroup of the group $\text{Aut}(X)$ of simplicial automorphisms of X (in the compact-open topology). If this is not the case, one can pass to the quotient of G by the kernel of the G –action on X (that quotient is a subgroup of $\text{Aut}(X)$), and then take its closure in $\text{Aut}(X)$.

Let $L^2 C^i(X)$ be the space of i –cochains on X which are square-summable with respect to the counting measure on the set $X^{(i)}$ of i –simplices in X . Then the coboundary map $\delta^i: L^2 C^i(X) \rightarrow L^2 C^{i+1}(X)$ is a bounded operator. The reduced L^2 –cohomology of X is defined to be $L^2 H^i(X) = \ker \delta^i / \overline{\text{im} \delta^{i-1}}$. This is a Hilbert space, carrying a unitary G –representation. Using the von Neumann G –dimension one defines $L^2 b^i(X) = \dim_G L^2 H^i(X)$. We are interested in calculating these Betti numbers. (This problem was considered by Dymara and Januszkiwicz in [8] and by Davis and Okun in [6].)

The first step is to pass from the cochain complex $(L^2 C^*(X), \delta)$ to a smaller complex of B –invariants: $(L^2 C^*(X)^B, \delta)$. Now $L^2 C^i(X)^B$ can be identified with a space of cochains

on $X/B = \Sigma$ —the Davis complex of the Weyl group W of the building. However, a simplex $\sigma \in \Sigma$ has a preimage in X consisting of $q^{d(\sigma)}$ simplices, where $q + 1$ is the thickness of the building and $d(\sigma)$ is the distance from σ to the chamber stabilised by B . Therefore a cochain f on Σ represents a square-summable B –invariant cochain if and only if it satisfies $\sum_{\sigma} |f(\sigma)|^2 q^{d(\sigma)} < \infty$; we denote the space of such cochains $L_q^2 C^*(\Sigma)$. The complex $(L_q^2 C^*(\Sigma), \delta)$ and its (reduced) cohomology $L_q^2 H^*(\Sigma)$ are acted upon by the Hecke algebra $\mathbf{C}[B \backslash G/B]$. A suitable von Neumann completion of the latter can be used to measure the dimension of $L_q^2 H^i(\Sigma)$, yielding Betti numbers $L_q^2 b^i(\Sigma)$. It turns out that $L_q^2 b^i(\Sigma) = L^2 b^i(X)$. In particular, the L^2 –Betti numbers of a building depend only on W and on q .

The good news is that the complex $(L_q^2(\Sigma), \delta)$, the Hecke algebra and the Betti numbers $L_q^2 b^i(\Sigma)$ can be defined for all real $q > 0$, in a uniform manner which for integer values of q gives exactly the objects discussed above. It turns out that for small q (namely for $q < \rho_W$, where ρ_W is the logarithmic growth rate of W) the Betti numbers $L_q^2 b^i(\Sigma)$ are 0 except for $i = 0$. Since $\rho_W \leq 1$, this result says nothing about actual buildings. However, in [Section 6](#) we prove a version of Poincaré duality, saying that if Σ is a manifold of dimension n , then $L_q^2 b^i(\Sigma) = L_{1/q}^2 b^{n-i}(\Sigma)$. Thus, if the Davis complex of the Weyl group of a building (ie, an apartment in the Davis realization of the building) is an n –manifold, and if $q > \frac{1}{\rho_W}$, then the L^2 –Betti numbers of the building vanish except for $L^2 b^n(X)$.

Examples of buildings to which our method applies can be constructed from flag triangulations of spheres. Davis associates a right-angled Coxeter group to any such triangulation; this right-angled Coxeter group is the Weyl group of a family of buildings with manifold apartments, parametrised by thickness. Let us mention that the argument applies also to Euclidean buildings, yielding another calculation of their L^2 –Betti numbers.

In a forthcoming paper (Davis–Dymara–Januszkiwicz–Okun [\[5\]](#)) the L^2 –Betti numbers of all buildings satisfying $q > \frac{1}{\rho_W}$ are calculated.

The definitions, results and arguments of this paper go through, with appropriate reading, in the multi-parameter case. A detailed account of the multi-parameter setting is given in [\[5\]](#).

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0 Integer thickness

Let (W, S) be a Coxeter system. Let Δ be a simplex with codimension 1 faces labelled by elements of S , and let Δ' be its first barycentric subdivision. Each $T \subseteq S$ generates a subgroup W_T of W called a special subgroup; also, T corresponds to a face Δ_T of Δ (the intersection of codimension 1 faces labelled by elements of T). The Davis chamber D is the subcomplex of Δ' spanned by barycentres of faces Δ_T for which W_T is finite (\mathcal{F} will denote the set of subsets $T \subseteq S$ such that W_T is finite). To every $T \subseteq S$ we assign a face of the Davis chamber: $D_T = D \cap \Delta_T$. The Davis realization Σ of the Coxeter complex is $W \times D / \sim$, where $(w, p) \sim (u, q)$ if and only if for some T we have $p = q \in D_T$ and $w^{-1}u \in W_T$. The action of W on the first factor descends to an action on Σ . We denote the image of σ under the action of w by $w\sigma$, and the W -orbit of σ in Σ by $W\sigma$. The images of $w \times D$ in Σ are called chambers. The action of W on Σ is simply transitive on the set of chambers.

A Tits building X_{Tits} with Weyl group W is a set with a W -valued distance function d , satisfying certain conditions (see Ronan [11]). Its Davis incarnation is $X = X_{Tits} \times D / \sim$, where $(x, p) \sim (y, q)$ if and only if for some T we have $p = q \in D_T$ and $d(x, y) \in W_T$. The images of $x \times D$ in X are called chambers.

We will consider only buildings of uniformly bounded thickness, ie, such that for some constant $N > 0$, any $s \in S$ and any $x \in X_{Tits}$ there are no more than N elements $y \in X_{Tits}$ satisfying $d(x, y) = s$. If this number of s -neighbours of x is equal to q for all pairs (x, s) , then we say that the building has uniform thickness $q + 1$. We denote such building $X(q)$ (for a right-angled Coxeter group it is unique).

Uniformly bounded thickness is equivalent to X being uniformly locally finite. Thus we can consider (reduced) L^2 -(co)homology of X . This is obtained from the complex of L^2 (co)chains on X with the usual (co)boundary operators ∂, δ . These operators are in fact adjoint to each other, so that the (co)homology can be identified with $L^2\mathcal{H}^*(X)$, the space of harmonic (co)chains (“reduced” means that we divide the kernel by the closure of the image).

Assume now that X_{Tits} comes from a BN -pair in a group G . Then G acts by simplicial automorphisms on X . We can assume that G acts faithfully and is locally compact (possibly taking the closure of its image in $\text{Aut}(X)$ in the compact-open topology). We use G to measure the size of $L^2\mathcal{H}^i(X)$ via the von Neumann dimension. To do this, we first express $L^2C^i(X)$ as $\bigoplus_{\sigma^i \subset D} L^2(G\sigma^i)$. Then we notice that $L^2(G\sigma^i)$ is naturally isomorphic to $L^2(G)^{G_{\sigma^i}}$ (where G_{σ^i} is the stabiliser of σ^i in G). It is convenient to multiply this isomorphism by a suitable scalar factor in order to make it isometric. Then

the space $L^2(G)^{G_{\sigma^i}}$ is embedded into $L^2(G)$, giving us finally an embedding of left G –modules $L^2C^i(X) \hookrightarrow \bigoplus_{\sigma^i \subset D} L^2(G)$. In particular, $L^2\mathcal{H}^i(X)$ is now embedded as a left G –module in $\bigoplus_{\sigma^i \subset D} L^2(G)$; we can consider the orthogonal projection onto this subspace, and define $L^2b^i(X)$ to be the von Neumann trace of that projection. Let B be the stabiliser of D in G . For each $\sigma^i \subset D$ we have a vector $\mathbf{1}_\sigma$ in $\bigoplus_{\sigma^i \subset D} L^2(G)$, having σ th component $\mathbf{1}_B$ and other components 0. The projection onto $L^2\mathcal{H}^i(X)$ is given by a matrix whose σ th row gives the projection of $\mathbf{1}_\sigma$ on $L^2\mathcal{H}^i(X)$, expressed as an element of $\bigoplus_{\sigma^i \subset D} L^2(G)$ (while applying this matrix we understand multiplication as convolution). Notice that both $\mathbf{1}_\sigma$ and the space $L^2\mathcal{H}^i(X)$ are B –invariant; so therefore will be the projection of $\mathbf{1}_\sigma$ on $L^2\mathcal{H}^i(X)$.

1 Real thickness

For a $w \in W$ we denote by $d(w)$ the length of a shortest word in the generators S representing w . For a chamber $c = w \times D$ of Σ we put $d(c) = d(w)$. For every simplex $\sigma \subset \Sigma$ there is a unique chamber $c \supseteq \sigma$ with smallest $d(c)$; we put $d(\sigma) = d(c)$.

For a real number $t > 0$ we equip the set $\Sigma^{(i)}$ of i –simplices in Σ with the measure $\mu_t(\sigma) = t^{d(\sigma)}$. We also pick (arbitrarily) orientations of simplices in D , and extend them W –equivariantly to orientations of all simplices in Σ . This allows us to identify chains and cochains with functions. We put

$$L_t^2 C^i(\Sigma) = L_t^2 C_i(\Sigma) = L^2(\Sigma^{(i)}, \mu_t).$$

We now define $\delta^i: L_t^2 C^i(\Sigma) \rightarrow L_t^2 C^{i+1}(\Sigma)$ by

$$\delta^i(f)(\tau^{i+1}) = \sum_{\sigma^i \subset \tau^{i+1}} [\tau : \sigma] f(\sigma)$$

and $\partial_i^t: L_t^2 C_i(\Sigma) \rightarrow L_t^2 C_{i-1}(\Sigma)$ by

$$\partial_i^t(f)(\eta^{i-1}) = \sum_{\sigma^i \supset \eta^{i-1}} [\eta : \sigma] t^{d(\sigma) - d(\eta)} f(\sigma)$$

(here $[\alpha : \beta] = \pm 1$ tells us whether orientations of α and β agree or not). We have

$$\begin{aligned} \langle \delta^i(f), g \rangle_t &= \sum_{\tau^{i+1}} \left(\sum_{\sigma^i \subset \tau^{i+1}} [\tau : \sigma] f(\sigma) \overline{g(\tau)} t^{d(\tau)} \right) \\ &= \sum_{\sigma^i} f(\sigma) \overline{\left(\sum_{\tau^{i+1} \supset \sigma^i} [\tau : \sigma] t^{d(\tau) - d(\sigma)} g(\tau) \right)} t^{d(\sigma)} = \langle f, \partial_i^t(g) \rangle_t. \end{aligned}$$

That is, $\delta^* = \partial^t$ as operators on $L_t^2 C^*(\Sigma)$. It follows that $(\partial^t)^2 = 0$ (since $\delta^2 = 0$), and we can consider (reduced) L_t^2 –(co)homology:

$$L_t^2 H^i(\Sigma) = \ker \delta^i / \overline{\text{im } \delta^{i-1}}, \quad L_t^2 H_i(\Sigma) = \ker \partial_i^t / \overline{\text{im } \partial_{i+1}^t}$$

Since $\delta^* = \partial^t$, $(\partial^t)^* = \delta$ we have $L_t^2 C^i(\Sigma) = \ker \partial_i^t \oplus \overline{\text{im } \delta^{i-1}} = \ker \delta^i \oplus \overline{\text{im } \partial_{i+1}^t}$ (orthogonal direct sums). It follows that

$$L_t^2 H^i(\Sigma) \simeq L_t^2 \mathcal{H}^i(\Sigma) \simeq L_t^2 H_i(\Sigma),$$

where $L_t^2 \mathcal{H}^i(\Sigma)$ is the space $\ker \delta^i \cap \ker \partial_i^t$ of harmonic i –cochains.

Remark Suppose that $X(q)$ is a building associated to a BN –pair, with Weyl group W . Then the B –invariant part of the L^2 cochain complex of $X(q)$ is isomorphic to $L_q^2 C^*(\Sigma)$.

2 Hecke algebra

We deform the usual scalar product on $\mathbf{C}[W]$ into $\langle \cdot, \cdot \rangle_t$:

$$(2-1) \quad \left\langle \sum_{w \in W} a_w \delta_w, \sum_{w \in W} b_w \delta_w \right\rangle_t = \sum_{w \in W} a_w \overline{b_w} t^{d(w)}.$$

We also correspondingly deform the multiplication into the following Hecke t –multiplication: for $w \in W$, $s \in S$ we put

$$(2-2) \quad \delta_w \delta_s = \begin{cases} \delta_{ws} & \text{if } d(ws) > d(w); \\ t \delta_{ws} + (t-1) \delta_w & \text{if } d(ws) < d(w). \end{cases}$$

This extends to a \mathbf{C} –bilinear associative multiplication on $\mathbf{C}[W]$ (see Bourbaki [1]). Using (2–2) and induction on $d(v)$ one easily shows

$$(2-3) \quad \delta_w \delta_v = \delta_{wv} \quad \text{if } d(wv) = d(w) + d(v).$$

We keep the involution on $\mathbf{C}[W]$ independent of t :

$$(2-4) \quad \left(\sum_{w \in W} a_w \delta_w \right)^* = \sum_{w \in W} \overline{a_{w^{-1}}} \delta_w.$$

Proposition 2.1 *The above scalar product, multiplication and involution define a Hilbert algebra structure on $\mathbf{C}[W]$ (in the sense of Dixmier [7, A.54]); we use the notation $\mathbf{C}_t[W]$ to indicate this structure.*

Proof We begin with involutivity: $(xy)^* = y^*x^*$. One checks it using (2–2) and (2–3) for $x = \delta_w$, $y = \delta_s$ considering two cases: $d(ws) < d(w)$, $d(ws) > d(w)$. Then one checks it for $x = \delta_w$, $y = \delta_u$ by induction on $d(u)$. Finally, by \mathbf{C} –bilinearity of multiplication, the result extends to general x, y . From involutivity and (2–2) we immediately get

$$(2-5) \quad \delta_s \delta_w = \begin{cases} \delta_{sw} & \text{if } d(sw) > d(w); \\ t\delta_{sw} + (t-1)\delta_w & \text{if } d(sw) < d(w). \end{cases}$$

We now recall and prove the conditions (i)–(iv) of [7] defining a Hilbert algebra.

$$(i) \quad \langle x, y \rangle_t = \langle y^*, x^* \rangle_t.$$

This is a straightforward calculation (using $d(w) = d(w^{-1})$).

$$(ii) \quad \langle xy, z \rangle_t = \langle y, x^*z \rangle_t.$$

Due to linearity it is enough to check (ii) in the case $y = \delta_w$, $z = \delta_u$, $x = \delta_v$. First one treats the case $v = s \in S$, directly using (2–5); this requires four sub-cases, depending on comparison of $d(sw)$ with $d(w)$ and $d(su)$ with $d(u)$. Then one performs an easy induction on $d(v)$.

$$(iii) \quad \text{For every } x \in \mathbf{C}_t[W] \text{ the map } \mathbf{C}_t[W] \ni y \mapsto xy \in \mathbf{C}_t[W] \text{ is continuous.}$$

One checks first that $y \mapsto \delta_s y$ is continuous, directly using (2–5). Continuity of $y \mapsto xy$ for arbitrary $x \in \mathbf{C}_t[W]$ follows, because compositions and linear combinations of continuous maps are continuous.

$$(iv) \quad \text{The set } \{xy \mid x, y \in \mathbf{C}_t[W]\} \text{ is dense in } \mathbf{C}_t[W].$$

This is immediate, since we have a unit element δ_1 in $\mathbf{C}_t[W]$. □

Corollary 2.2 *The coefficient of δ_1 in ab is equal to $\langle a, b^* \rangle_t$.*

Proof That coefficient is equal to $\langle ab, \delta_1 \rangle_t$, which by (ii) and (i) is $\langle b, a^* \rangle_t = \langle a, b^* \rangle_t$. □

As in [7, A.54], we get two von Neumann algebras U_t , V_t : they are weak closures of $\mathbf{C}_t[W]$ acting on its completion L_t^2 by left (respectively right) multiplication.

As in [7, A.57], we put $\mathbf{C}_t[W]'$ to be the algebra of all bounded elements of L_t^2 ; bounded means that left (or, equivalently, right) multiplication by the element is bounded on $\mathbf{C}_t[W]$ (so, extends to a bounded operator on L_t^2 and defines an element of U_t or V_t).

As in [7, A.60], we have natural traces tr on U_t, V_t : if $B \in U_t$ (or $B \in V_t$) is self-adjoint and positive, we ask whether $B^{\frac{1}{2}} = a \cdot$ (resp. $B^{\frac{1}{2}} = \cdot a$) for an $a \in \mathbf{C}_t[W]'$. If it is so, we put $\text{tr } B = \|a\|_t^2$; otherwise we put $\text{tr } B = +\infty$. The $a = \sum_{w \in W} a_w \delta_w$ we are asking for is self-adjoint: $a_w = \overline{a_{w^{-1}}}$, so that by Corollary 2.2 $\|a\|_t^2$ is equal to the coefficient of δ_1 in a^2 . Thus B is the multiplication by the bounded self-adjoint element $b = a^2$, and $\text{tr } B$ is equal to the coefficient of δ_1 in b .

Suppose now that we are given a closed subspace Z of $\bigoplus_{i=1}^l L_t^2$, such that the orthogonal projection P_Z onto Z is an element of $M_{l \times l} \otimes V_t$. To calculate the trace of this projection we first need to identify P_Z as a matrix. So, we take the standard basis $\{e_i\}$ of $\bigoplus_{i=1}^l L_t^2$ (e_i has δ_1 as the i th coordinate, and other coordinates 0), and apply P_Z to it. We expand the results in the basis $\{e_i\}$: let $a_i^j \in L_t^2$ be the j th coordinate of $P_Z(e_i)$. Then we take the coefficient of δ_1 in a_i^j and sum over i . The number we get is the trace of P_Z .

3 L_t^2 -Betti numbers

It will be convenient to identify L_t^2 with $L^2(W, \nu_t)$, where $\nu_t(w) = t^{d(w)}$. For any Coxeter group Γ (we have W as well as its subgroups W_T in mind) the generating function of Γ is defined by $\Gamma(x) = \sum_{\gamma \in \Gamma} x^{d(\gamma)}$. For a finite Γ it is a polynomial, in general it is a rational function. We denote by ρ_Γ the radius of convergence of the series defining $\Gamma(x)$.

As in the case of buildings (Section 0), we have $L_t^2 C^i(\Sigma) = \bigoplus_{\sigma^i \subseteq D} L^2(W\sigma^i, \mu_t)$. Now $L^2(W\sigma^i, \mu_t)$ can be identified with $L^2(W, \nu_t)^{W_{T(\sigma)}}$ (where $T(\sigma)$ is the largest subset of S such that $\sigma \subseteq D_{T(\sigma)}$) via the map ϕ given by $\phi(f)(w) = \frac{1}{\sqrt{W_{T(\sigma)}(t)}} f(w\sigma)$ (we distorted the natural map by the factor $\frac{1}{\sqrt{W_{T(\sigma)}(t)}}$ in order to make it isometric). Finally, $L^2(W, \nu_t)^{W_{T(\sigma)}}$ is a subspace of $L^2(W, \nu_t) = L_t^2$, so that we get an isometric embedding

$$\Phi: L_t^2 C^i(\Sigma) \hookrightarrow \bigoplus_{\sigma^i \subseteq D} L_t^2 = C^i(D) \otimes L_t^2.$$

Let \mathcal{L} denote the algebra U_t acting diagonally on the left on $\bigoplus_{\sigma \subseteq D} L_t^2 = C^*(D) \otimes L_t^2$; let \mathcal{R} be $\text{End}(C^*(D)) \otimes V_t$ acting on the same space on the right. The von Neumann algebras \mathcal{L} and \mathcal{R} are commutants of each other.

Lemma 3.1 *The projection of L_t^2 onto $L^2(W\sigma, \mu_t) = L^2(W, \nu_t)^{W_{T(\sigma)}}$ is given by the right Hecke t -multiplication by*

$$(3-1) \quad p_{T(\sigma)} = \frac{1}{W_{T(\sigma)}(t)} \sum_{w \in W_{T(\sigma)}} \delta_w.$$

Proof Put $T = T(\sigma)$. The subspace onto which we project consists of those elements of L_t^2 which are right W_T -invariant; this is equivalent to being invariant under right Hecke t -multiplication by $\frac{1}{1+t}(\delta_1 + \delta_s)$ for all $s \in T$ (to check this one splits W into pairs $\{w, ws\}$, and calculates for each pair separately using (2–2)). As a result, this subspace is \mathcal{L} -invariant, so that the projection P_T onto it is an element of \mathcal{R} . It follows that P_T is given by right Hecke t -multiplication by $P_T(\delta_1)$. The latter is clearly of the form $C \sum_{w \in W_T} \delta_w$, where C is a constant such that

$$\langle \delta_1 - C \sum_{w \in W_T} \delta_w, C \sum_{w \in W_T} \delta_w \rangle_t = 0.$$

This gives $C = \|\sum_{w \in W_T} \delta_w\|_t^{-2} = (\sum_{w \in W_T} t^{d(w)})^{-1} = \frac{1}{W_T(t)}$. \square

Lemma 3.2 \mathcal{L} preserves the subspace $L_t^2 C^i(\Sigma) \subset C^*(D) \otimes L_t^2$ and commutes with δ and ∂^t .

Proof The first claim follows from Lemma 3.1 (and actually was a step in the proof of that lemma). To prove the second part notice that δ is an element of \mathcal{R} : the matrix with V_t -coefficients describing δ has non-zero $\sigma\tau$ -entry if and only if σ is a codimension 1 face of τ ; the entry is then $\sqrt{\frac{W_{T(\sigma)}(t)}{W_{T(\tau)}(t)}} \delta_1$. It follows that δ commutes with \mathcal{L} . So therefore does its adjoint ∂^t . \square

Corollary 3.3 $L_t^2 C^i(\Sigma)$, $L_t^2 \mathcal{H}^i(\Sigma)$, $\ker \delta^i$, $\ker \partial_t^i$, $\overline{\text{im } \delta^i}$, $\overline{\text{im } \partial_t^i}$ are \mathcal{L} -invariant; therefore, orthogonal projections onto these spaces belong to \mathcal{R} .

We use tr to denote the tensor product of the usual matrix trace on $\text{End}(C^*(D))$ and the von Neumann trace on V_t as described in Section 2. We put

$$(3-2) \quad b_t^i = L_t^2 b^i(\Sigma) = \text{tr}(\text{projection onto } L_t^2 \mathcal{H}^i(\Sigma))$$

$$(3-3) \quad c_t^i = L_t^2 c^i(\Sigma) = \text{tr}(\text{projection onto } L_t^2 C^i(\Sigma))$$

$$(3-4) \quad \chi_t = \sum_i (-1)^i b_t^i = \sum_i (-1)^i c_t^i.$$

The sums in (3–4) give the same value by the standard algebraic topology argument. It follows from Lemma 3.1 that $c_t^i = \sum_{\sigma^i \subset D} \frac{1}{W_{T(\sigma)}(t)}$. Grouping together simplices σ with the same $T(\sigma)$ and using formula (5) from Charney–Davis [3] we obtain the following result (see Serre [12]).

Corollary 3.4

$$\chi_t = \frac{1}{W(t)}$$

Theorem 3.5 Suppose that $X(q)$ is a building associated to a BN-pair, with Weyl group W . Then $L^2 b^i(X(q)) = b_q^i$.

Proof For $t = q$, $L_t^2 C^i(\Sigma)$ coincides with the space of B -invariant elements of $L^2 C^i(X(q))$. By the concluding remarks of [Section 0](#), the matrix of the projection onto $L^2 \mathcal{H}^i(X(q))$ has B -invariant entries—so that it coincides with the one we use to define b_t^i . Hence the conclusion. \square

Suppose now that the pair $(D, \partial D = D \cap \partial \Delta)$ is a generalised homology n -disc (ie, it is a homology manifold with boundary, with relative homology groups the same as those of an n -disc modulo its boundary). Then each $D_T = D \cap \Delta_T$ is also a homology $(n - |T|)$ -disc (for $T \in \mathcal{F}$). We can now use wD_T , $w \in W$, $T \in \mathcal{F}$, as a homology cellular structure on Σ (denoted Σ_{ghd}). The cell D_T has the form of an o_T -centred cone; we put $d(wD_T) = d(wo_T)$, and define μ_t , (co)chain complexes, the embedding Φ , the U_t -module structure and the numbers $b_t^i(\Sigma_{ghd})$ in essentially the same way as for the original triangulation of Σ .

4 Dual cells

So far we used the triangulation of Σ which originated from the barycentric subdivision of a simplex. We will use notation Σ_{st} to remind that we have this standard triangulation in mind. In this section we will describe another cell structure on Σ . It will make our discussion of Poincaré duality in [Section 6](#) look pretty standard.

To each $T \in \mathcal{F}$ we associate a face Δ_T of Δ , whose barycentre o_T is a vertex of the Davis chamber D . We define $\langle T \rangle$ as the union of all simplices $\sigma \subset \Sigma$ such that $\sigma \cap D_T = o_T$ (recall that $D_T = D \cap \Delta_T$). As a simplicial complex, $\langle T \rangle$ is an o_T -centred cone over Σ_T ; since T is such that W_T is finite, Σ_T is a sphere and $\langle T \rangle$ is a disc of dimension $|T|$. The boundary of $\langle T \rangle$ is cellulated by $w\langle U \rangle$, for all possible $T \subset U \subseteq S$, $w \in W_T$. The complex Σ cellulated by $w\langle T \rangle$, over all $w \in W$, $T \in \mathcal{F}$, is a cellular complex that we denote Σ_d . The cells of Σ_d will be called *dual cells*. The name *Coxeter blocks* is also used (Davis [4]).

We now put $d(w\langle T \rangle) = d(wo_T)$, and define the measures μ_t on the set $\Sigma_d^{(i)}$ of i -dimensional cells of Σ_d by $\mu_t(\langle a \rangle) = t^{d(\langle a \rangle)}$. Then

$$L_t^2 C^i(\Sigma_d) = L_t^2 C_i(\Sigma_d) \simeq L^2(\Sigma_d^{(i)}, \mu_t),$$

We now define $\delta^i: L_t^2 C^i(\Sigma_d) \rightarrow L_t^2 C^{i+1}(\Sigma_d)$ by

$$\delta^i(f)(\langle \tau \rangle^{i+1}) = \sum_{\langle \sigma \rangle^i \subset \langle \tau \rangle^{i+1}} [\langle \tau \rangle : \langle \sigma \rangle] f(\langle \sigma \rangle)$$

and $\partial_i^t: L_t^2 C_i(\Sigma_d) \rightarrow L_t^2 C_{i-1}(\Sigma_d)$ by

$$\partial_i^t(f)(\langle \eta \rangle^{i-1}) = \sum_{\langle \sigma \rangle^i \supset \langle \eta \rangle^{i-1}} [\langle \eta \rangle : \langle \sigma \rangle] t^{d(\langle \sigma \rangle) - d(\langle \eta \rangle)} f(\langle \sigma \rangle).$$

The discussion from [Section 1](#) can be continued, and supplies us with $L_t^2 \mathcal{H}^i(\Sigma_d)$. Now we wish to bring in the Hecke algebra. We pick (arbitrarily) orientations of the cells $\langle T \rangle$ ($T \in \mathcal{F}$), and extend these to orientations of all cells in Σ_d as follows: $w\langle T \rangle$ is the oriented cell which is the image of the oriented cell $\langle T \rangle$ by w , with orientation changed by a factor of $(-1)^{d(w)}$. Using these orientations, we identify $L_t^2 C^*(\Sigma_d)$ with $\bigoplus_{T \in \mathcal{F}} L^2(W\langle T \rangle, \mu_t)$. For every $T \in \mathcal{F}$ we define a map $\psi_T: L^2(W\langle T \rangle, \mu_t) \rightarrow L_t^2$ by the formula

$$(4-1) \quad \psi_T(f) = \sum_{w \in W^T} f(w\langle T \rangle) (-1)^{d(w)} \sqrt{W_T(t^{-1})} \delta_w h_T,$$

where $W^T = \{w \in W \mid \forall u \in W_T, d(wu) \geq d(w)\}$ (the set of T -reduced elements), and

$$(4-2) \quad h_T = \frac{1}{W_T(t^{-1})} \sum_{u \in W_T} (-t)^{-d(u)} \delta_u.$$

Putting together these maps we get a map $\Psi: L_t^2 C^*(\Sigma_d) \rightarrow \bigoplus_{T \in \mathcal{F}} L_t^2$.

Lemma 4.1 (1) For all $s \in T$ we have $\delta_s h_T = -h_T$.

(2) For all $u \in W_T$ we have $\delta_u h_T = (-1)^{d(u)} h_T$.

(3) For all $U \subseteq T$ we have $h_U h_T = h_T$.

Proof (1) Let $w \in W$ be such that $d(sw) > d(w)$. Then $\delta_s \delta_w = \delta_{sw}$ (by [\(2-3\)](#)). We then have

$$\begin{aligned} \delta_s(\delta_w - \frac{1}{t} \delta_{sw}) &= \delta_{sw} - \frac{1}{t} (\delta_s \delta_w) \delta_w = \delta_{sw} - \frac{1}{t} (t \delta_1 + (t-1) \delta_s) \delta_w \\ &= (1 - \frac{t-1}{t}) \delta_{sw} - \delta_w = -(\delta_w - \frac{1}{t} \delta_{sw}) \end{aligned}$$

Since h_T is a linear combination of expressions of the form $\delta_w - \frac{1}{t} \delta_{sw}$, (1) follows.

(2) Follows from (1) by induction on $d(u)$.

$$\begin{aligned}
(3) \quad h_U h_T &= \frac{1}{W_U(t^{-1})} \sum_{u \in W_U} (-t)^{-d(u)} \delta_u h_T \\
&= \frac{1}{W_U(t^{-1})} \sum_{u \in W_U} (-t)^{-d(u)} (-1)^{d(u)} h_T \\
&= \frac{1}{W_U(t^{-1})} \left(\sum_{u \in W_U} t^{-d(u)} \right) h_T = h_T
\end{aligned}
\quad \square$$

Lemma 4.2 (1) For every $T \in \mathcal{F}$ the map ψ_T is an isometric embedding.

(2) The orthogonal projection of L_t^2 onto the image of ψ_T is given by right Hecke t -multiplication by h_T .

Proof (1) The squared norm of a summand from the right hand side of (4–1) is

$$\|f(w\langle T \rangle)(-1)^{d(w)} \sqrt{W_T(t^{-1})} \delta_w h_T\|_t^2 = |f(w\langle T \rangle)|^2 W_T(t^{-1}) \|\delta_w h_T\|_t^2.$$

Since w is T -reduced, we have $\delta_w \delta_u = \delta_{wu}$ for all $u \in W_T$. Therefore

$$\begin{aligned}
\|\delta_w h_T\|_t^2 &= \left\| \frac{1}{W_T(t^{-1})} \sum_{u \in W_T} (-t)^{-d(u)} \delta_{wu} \right\|_t^2 = \left| \frac{1}{W_T(t^{-1})} \right|^2 \sum_{u \in W_T} | -t |^{-2d(u)} t^{d(wu)} \\
&= t^{d(w)} \frac{1}{W_T(t^{-1})^2} \sum_{u \in W_T} t^{-d(u)} = t^{d(w)} \frac{1}{W_T(t^{-1})}.
\end{aligned}$$

(2) Due to $h_T h_T = h_T$ and $h_T^* = h_T$, right Hecke t -multiplication by h_T is an orthogonal projection. Let $w \in W$; write $w = vu$ where $u \in W_T$ and v is T -reduced. Then $\delta_w h_T = \delta_v \delta_u h_T = (-1)^{d(u)} \delta_v h_T$. This shows that image of the space of finitely supported functions (on $W\langle T \rangle$) under ψ_T is equal to the image of the space of finitely supported functions (on W) under right Hecke t -multiplication by h_T . Since ψ_T is isometric, the L_t^2 -completions of these images also coincide. \square

Denote by \mathcal{L} the algebra U_t acting diagonally on the left on $\bigoplus_{T \in \mathcal{F}} L_t^2$, and by \mathcal{R} its commutant $M_{|\mathcal{F}|}(\mathbb{C}) \otimes V_t$ (acting on the right). It follows from Lemma 4.2 that the image of Ψ is \mathcal{L} -invariant. In other words, we have a U_t -module structure on $L_t^2 C^*(\Sigma_d)$, defined by the condition that the isometric embedding $\Psi: L_t^2 C^*(\Sigma_d) \rightarrow \bigoplus L_t^2$ is a morphism of U_t -modules. Thus, we think of $L_t^2 C^*(\Sigma_d)$ as of a submodule of $\bigoplus_{T \in \mathcal{F}} L_t^2$.

Lemma 4.3 *The map $\delta: L_t^2 C^*(\Sigma_d) \rightarrow L_t^2 C^*(\Sigma_d)$ is (a restriction of) an element of \mathcal{R} . For $U \subset T \in \mathcal{F}$ satisfying $|T| = |U| + 1$, the UT -entry of this element is*

$$[\langle T \rangle : \langle U \rangle] \sqrt{\frac{W_T(t^{-1})}{W_U(t^{-1})}} h_T$$

Proof Consider a pair of cells $w\langle U \rangle, w\langle T \rangle$. We have $[w\langle T \rangle : w\langle U \rangle] = [\langle T \rangle : \langle U \rangle]$. We can assume that w is U -reduced, and write it as vu , where v is T -reduced and $u \in W_T$. Let $f \in L_t^2 C^{\dim\langle U \rangle}(\Sigma_d)$. The summand in $\psi_U(f)$ corresponding to the cell $w\langle U \rangle$ is

$$f(w\langle U \rangle)(-1)^{d(w)} \sqrt{W_U(t^{-1})} \delta_w h_U.$$

The summand in $\psi_T(\delta f)$ corresponding to the contribution of $f(w\langle U \rangle)$ to $(\delta f)(w\langle T \rangle)$ is

$$[\langle T \rangle : \langle U \rangle] f(w\langle U \rangle)(-1)^{d(v)} \sqrt{W_T(t^{-1})} \delta_v h_T.$$

Now $\delta_w h_U h_T = \delta_w h_T = \delta_v \delta_u h_T = (-1)^{d(u)} \delta_v h_T$, and the lemma follows. \square

Corollary 4.4 *The subspaces $L_t^2 C^i(\Sigma_d)$, $L_t^2 \mathcal{H}^i(\Sigma_d)$, $\ker \delta^i$, $\ker \partial_t^i$, $\overline{\text{im } \delta^i}$ and $\overline{\text{im } \partial_t^i}$ of $\bigoplus_{T \in \mathcal{F}} L_t^2$ are \mathcal{L} -invariant; therefore, orthogonal projections onto these spaces are elements of \mathcal{R} .*

5 Invariance

In this section we prove that $L_t^2 H^*(\Sigma_d) \simeq L_t^2 H^*(\Sigma_{st})$ ($\simeq L_t^2 H^*(\Sigma_{ghd})$, if the latter exists) as U_t -modules. It will be convenient for us to work with homology rather than cohomology; since both are isomorphic to the U_t -module of harmonic cochains, it makes no difference.

We start by fixing orientation conventions. Let us pick arbitrary orientations of the dual cells $\langle T \rangle$ for all $T \in \mathcal{F}$. We extend these orientations to all dual cells as in [Section 4](#) ($w\langle T \rangle$ is oriented by $(-1)^{d(w)}$ times the orientation of $\langle T \rangle$ pushed forward by w). For $T \in \mathcal{F}$ of cardinality k , let $\langle T \rangle \cap D^{(k)}$ be the set of all k -simplices of Σ_{st} contained in $\langle T \rangle \cap D$. We orient every element of $\langle T \rangle \cap D^{(k)}$ by the restriction of the chosen orientation of $\langle T \rangle$. We then extend these orientations W -equivariantly (to a part of Σ_{st}), and put arbitrary equivariant orientations on the rest of Σ_{st} . Notice that if a k -simplex σ is contained in $w\langle T \rangle$ (where T has cardinality k), then the orientation of σ agrees with $(-1)^{d(\sigma)}$ times that of $w\langle T \rangle$. Orientations being chosen, we treat (co)chains as functions on the set of cells/simplices.

We define a topological embedding of Hilbert spaces $\theta: L_t^2 C^*(\Sigma_d) \rightarrow L_t^2 C^*(\Sigma_{st})$.

Definition Let $f \in L_t^2 C^k(\Sigma_d)$, $\sigma \in \Sigma_{st}^{(k)}$.

(1) If there exists $\langle \alpha \rangle \in \Sigma_d^{(k)}$ such that $\sigma \subseteq \langle \alpha \rangle$ (there is at most one such $\langle \alpha \rangle$), then

$$\theta f(\sigma) = (-1)^{d(\sigma)} t^{d(\langle \alpha \rangle) - d(\sigma)} f(\langle \alpha \rangle).$$

(2) If there is no $\langle \alpha \rangle$ as in (1), we put $\theta f(\sigma) = 0$.

Lemma 5.1

$$\partial^t \theta = \theta \partial^t.$$

Proof We will show that for all $f \in L_t^2 C^k(\Sigma_d)$, $\sigma \in \Sigma_{st}^{(k)}$ we have $\partial^t \theta f(\sigma) = \theta \partial^t f(\sigma)$. There are two cases to consider.

(1) Suppose that there exists $\langle \alpha \rangle \in \Sigma_d^{(k)}$ such that $\sigma \subseteq \langle \alpha \rangle$. Then

$$\begin{aligned} \theta \partial^t f(\sigma) &= (-1)^{d(\sigma)} t^{d(\langle \alpha \rangle) - d(\sigma)} \partial^t f(\langle \alpha \rangle) \\ &= (-1)^{d(\sigma)} t^{d(\langle \alpha \rangle) - d(\sigma)} \sum_{\langle \beta \rangle^{k+1} \supset \langle \alpha \rangle} [\langle \beta \rangle : \langle \alpha \rangle] t^{d(\langle \beta \rangle) - d(\langle \alpha \rangle)} f(\langle \beta \rangle) \\ (5-1) \quad &= (-1)^{d(\sigma)} \sum_{\langle \beta \rangle^{k+1} \supset \langle \alpha \rangle} [\langle \beta \rangle : \langle \alpha \rangle] t^{d(\langle \beta \rangle) - d(\sigma)} f(\langle \beta \rangle). \end{aligned}$$

On the other hand,

$$(5-2) \quad \partial^t \theta f(\sigma) = \sum_{\tau^{k+1} \supset \sigma} [\tau : \sigma] t^{d(\tau) - d(\sigma)} \theta f(\tau).$$

Notice that if $\theta f(\tau) \neq 0$ then there exists a dual cell $\langle \beta \rangle^{k+1} \supset \tau$. Such $\langle \beta \rangle$ is unique and $\langle \tau \rangle$ is the only $(k+1)$ -simplex in $\langle \beta \rangle$ with face $\langle \sigma \rangle$. Therefore (5-2) equals

$$\begin{aligned} &\sum_{\langle \beta \rangle^{k+1} \supset \langle \alpha \rangle} [\tau : \sigma] t^{d(\tau) - d(\sigma)} (-1)^{d(\tau)} t^{d(\langle \beta \rangle) - d(\tau)} f(\langle \beta \rangle) \\ (5-3) \quad &= \sum_{\langle \beta \rangle^{k+1} \supset \langle \alpha \rangle} [\tau : \sigma] (-1)^{d(\tau)} t^{d(\langle \beta \rangle) - d(\sigma)} f(\langle \beta \rangle). \end{aligned}$$

Now (5-3) and (5-1) are equal because $[\tau : \sigma] = (-1)^{d(\tau)} (-1)^{d(\sigma)} [\langle \beta \rangle : \langle \alpha \rangle]$.

(2) The smallest dual cell $\langle \alpha \rangle$ containing σ is of dimension $m > k$. Then $\theta \partial^t f(\sigma) = 0$. On the other hand,

$$\partial^t \theta f(\sigma) = \sum_{\tau^{k+1} \supset \sigma} [\tau : \sigma] t^{d(\tau) - d(\sigma)} \theta f(\tau).$$

Let $\tau^{k+1} \supset \sigma$, and let $\langle \beta \rangle \supset \langle \alpha \rangle$ be the smallest dual cell containing τ . If $\theta f(\tau) \neq 0$, then $\dim \langle \beta \rangle = k+1$, which forces $\langle \beta \rangle = \langle \alpha \rangle$ and $m = \dim \langle \alpha \rangle = k+1$. Thus,

we are reduced to the case $m = k + 1$. In this case, there are exactly two simplices $\sigma_{\pm} \in \Sigma_{st}^{(k+1)}$, $\sigma_{\pm} \subset \langle \alpha \rangle$, $\sigma_{\pm} \supset \sigma$. Since σ_{\pm} is oriented by $(-1)^{d(\sigma_{\pm})}$ times the orientation of $\langle \alpha \rangle$, we have

$$(5-4) \quad (-1)^{d(\sigma_+)}[\sigma_+ : \sigma] = -(-1)^{d(\sigma_-)}[\sigma_- : \sigma].$$

Therefore

$$\begin{aligned} \partial^t \theta(\sigma) &= [\sigma_+ : \sigma] t^{d(\sigma_+) - d(\sigma)} \theta(\sigma_+) + [\sigma_- : \sigma] t^{d(\sigma_-) - d(\sigma)} \theta(\sigma_-) \\ &= [\sigma_+ : \sigma] t^{d(\sigma_+) - d(\sigma)} (-1)^{d(\sigma_+)} t^{d(\langle \alpha \rangle) - d(\sigma_+)} f(\langle \alpha \rangle) \\ &\quad + [\sigma_- : \sigma] t^{d(\sigma_-) - d(\sigma)} (-1)^{d(\sigma_-)} t^{d(\langle \alpha \rangle) - d(\sigma_-)} f(\langle \alpha \rangle) \\ &= ((-1)^{d(\sigma_+)}[\sigma_+ : \sigma] + (-1)^{d(\sigma_-)}[\sigma_- : \sigma]) t^{d(\langle \alpha \rangle)} f(\langle \alpha \rangle) \\ (5-5) \quad &= 0. \end{aligned} \quad \square$$

Lemma 5.2 θ is a morphism of U_t -modules.

Proof The U_t -module structures on $L_t^2 C^k(\Sigma_d)$ and on $L_t^2 C^k(\Sigma_{st})$ are defined via embeddings Ψ and Φ . We will compare Ψ and $\Phi \circ \theta$. Let $f \in L_t^2 C^k(\Sigma_d)$; $\Psi(f)$ is a collection of $\psi_T(f)$, where

$$(5-6) \quad \psi_T(f) = \sum_{w \in W^T} f(w\langle T \rangle) (-1)^{d(w)} \sqrt{W_T(t^{-1})} \delta_w h_T.$$

The part of θ corresponding to $\psi_T(f)$ is supported by the set of W -translates of simplices $\sigma \in \langle T \rangle \cap D^{(k)}$, and is mapped by Φ into $\bigoplus_{\sigma \in \langle T \rangle \cap D^{(k)}} L_t^2$. The component indexed by σ is $\sum_{w \in W} \theta(w\sigma) \delta_w$ (notice that the stabiliser of σ is trivial), ie,

$$(5-7) \quad \sum_{w \in W} (-1)^{d(w\langle T \rangle)} t^{d(w\langle T \rangle) - d(w\sigma)} f(w\langle T \rangle) \delta_w.$$

Comparing (5-6) and (5-7) with the help of (4-2), we get that $\psi_T(f)$ agrees with (every component of) the corresponding part of $\Phi(\theta)$, up to a multiplicative factor of $\sqrt{W_T(t^{-1})}$. This implies the lemma. \square

Theorem 5.3 The map θ induces an isomorphism of U_t -modules $L_t^2 H_*(\Sigma_d) \simeq L_t^2 H_*(\Sigma_{st})$.

Proof Lemmas 5.1 and 5.2 imply that θ induces a morphism of U_t -modules on homology. We have to check that it is an isomorphism of vector spaces.

Let K_* be the image of θ . It is a subcomplex of $(L_t^2 C_*(\Sigma_{st}), \partial^t)$. A k -chain $c \in L_t^2 C_*(\Sigma_{st})$ is in K_* if and only if the following two conditions hold:

- (1) c is supported by the union of k -dimensional dual cells: $\bigcup \Sigma_d^{(k)}$,
- (2) if $\sigma^k, \tau^k \subseteq \langle \alpha \rangle^k$, then $c(\sigma) = (-t)^{d(\tau)-d(\sigma)} c(\tau)$.

We need to show that the inclusion $K_* \hookrightarrow L_t^2 C_*(\Sigma_{st})$ induces an isomorphism on (reduced) homology.

Let $m_t: L_t^2 C_*(\Sigma_{st}) \rightarrow L_{t-1}^2 C_*(\Sigma_{st})$ be the isomorphism (of Hilbert spaces) $m_t f(\sigma) = t^{d(\sigma)} f(\sigma)$. Instead of working directly with K_* , $L_t^2 C_*(\Sigma_{st})$ and ∂^t , we will work with $L_* = m_t(K_*)$, $E_* = L_{t-1}^2 C_*(\Sigma_{st}) = m_t(L_t^2 C_*(\Sigma_{st}))$ and $\partial = m_t \partial^t m_t^{-1}$. The advantage is that

$$\begin{aligned}
 \partial g(\sigma) &= m_t \partial^t m_t^{-1} g(\sigma) = t^{d(\sigma)} \partial^t m_t^{-1} g(\sigma) \\
 (5-8) \quad &= t^{d(\sigma)} \sum_{\tau^{k+1} \supset \sigma} [\tau : \sigma] t^{d(\tau)-d(\sigma)} m_t^{-1} g(\tau) \\
 &= \sum_{\tau^{k+1} \supset \sigma} [\tau : \sigma] t^{d(\tau)} t^{-d(\tau)} g(\tau) = \sum_{\tau^{k+1} \supset \sigma} [\tau : \sigma] g(\tau).
 \end{aligned}$$

To check whether $c \in E_*$ is in L_* we use (1) and the following version of (2):

- (2') if $\sigma^k, \tau^k \subseteq \langle \alpha \rangle^k$, then $c(\sigma) = (-1)^{d(\tau)-d(\sigma)} c(\tau)$.

Lemma 5.4 *Let $c \in E_k$. If $\partial c \in L_*$, then there exists a $d \in E_{k+1}$ such that $c - \partial d \in L_*$. Moreover, there is a constant C depending only on W and t such that d can be chosen so that $\|d\| \leq C\|c\|$.*

Proof Each dual cell $\langle \alpha \rangle$ is a disc; we denote by $\text{int} \langle \alpha \rangle$ its interior, and by $\text{bd} \langle \alpha \rangle$ its boundary. We construct, by descending induction on m ($m \geq k$), cochains $d_m \in E_{k+1}$ such that $c - \partial d_m$ is supported by the union of dual cells of dimensions at most m . For $m \geq \dim \Sigma$ we put $d_m = 0$. Suppose that d_m is already constructed, where $m > k$. For every dual m -cell $\langle \alpha \rangle$, let c_α be the restriction of $c - \partial d_m$ to $\langle \alpha \rangle$ (ie, if $c - \partial d_m = \sum a_\sigma \sigma$, then $c_\alpha = \sum_{\sigma \subseteq \langle \alpha \rangle} a_\sigma \sigma$). Let $\sigma^k \cap \text{int} \langle \alpha \rangle \neq \emptyset$. Then σ appears in ∂c_α and in $\partial c = \partial(c - \partial d)$ with the same coefficient, due to the inductive assumption. But, since $\partial c \in L_*$, this coefficient is 0. As a result, $c_\alpha \in Z_k(\langle \alpha \rangle, \text{bd} \langle \alpha \rangle)$. Since $H_k(\langle \alpha \rangle, \text{bd} \langle \alpha \rangle) = 0$ (recall that $m = \dim \langle \alpha \rangle > k$), we can find $d_\alpha \in C_{k+1}(\langle \alpha \rangle)$ such that $c_\alpha - \partial d_\alpha \in C_k(\text{bd} \langle \alpha \rangle)$. Moreover, we can choose d_α so that $\|d_\alpha\| \leq C_1 \|c_\alpha\|$, for some constant C_1 depending only on W and t . Due to uniform local finiteness of Σ , we deduce $\| \sum_{\langle \alpha \rangle} d_\alpha \| \leq C_2 \|c\|$ for some constant C_2 . We put $d_{m-1} = d_m + \sum_{\langle \alpha \rangle \in \Sigma_d^{(m)}} d_\alpha$, and $d = d_k$.

The estimate $\|d\| \leq C\|c\|$ clearly follows from the construction. The chain $c - \partial d = \sum b_\sigma \sigma$ is supported by the union of dual cells of dimensions at most k . Let us check

that it satisfies the condition (2'). Suppose that $\sigma^{k-1} \cap \text{int}\langle\alpha\rangle^k \neq \emptyset$. There are exactly two k -simplices $\sigma_{\pm} \subset \langle\alpha\rangle$ such that $\sigma \subset \sigma_{\pm}$. The coefficient of σ in $\partial(c - \partial d) = \partial c$ is 0 (because $\partial c \in L_*$), and, on the other hand, is equal to $[\sigma_+ : \sigma]b_{\sigma_+} + [\sigma_- : \sigma]b_{\sigma_-}$. Using (5–4) we get $b_{\sigma_+} = (-1)^{d(\sigma_+)-d(\sigma_-)}b_{\sigma_-}$. This holds for all σ^{k-1} satisfying $\sigma^{k-1} \cap \text{int}\langle\alpha\rangle^k \neq \emptyset$, which implies that $c - \partial d$ satisfies (2'). Hence $c - \partial d \in L_*$. The lemma is proved. \square

We are ready to check that the inclusion $\iota: L_* \hookrightarrow E_*$ induces an isomorphism ι_* on (reduced) homology. To show that ι_* is surjective, suppose that $c \in E_*$ is closed: $\partial c = 0$. Then $\partial c \in L_*$, and, by Lemma 5.4, there exists $d \in E_*$ such that $c - \partial d \in L_*$. We get $[c] = \iota_*[c - \partial d]$.

To show that ι_* is 1–1, suppose that $l \in L_*$, $\partial l = 0$ and $\iota_*[l] = 0$, ie, $l = \lim \partial e_n$ for some sequence of $e_n \in E_*$. Applying Lemma 5.4 to $c = l - \partial e_n$, we get that there exist $f_n \in E_*$, $f_n \rightarrow 0$ such that $l - \partial e_n - \partial f_n \in L_*$. But, since $l \in L_*$, we deduce that $\partial(e_n + f_n) \in L_*$. Now we apply Lemma 5.4 to $c = e_n + f_n$ to get $g_n \in E_*$ such that $h_n = e_n + f_n - \partial g_n \in L_*$. We have

$$\partial h_n = \partial e_n + \partial f_n - \partial \partial g_n.$$

The last term is 0, the middle term converges to 0 since ∂ is bounded and $f_n \rightarrow 0$, so that, finally,

$$\lim \partial h_n = \lim \partial e_n = l.$$

This means that $[l] = 0$ in $H_*(L_*)$.

We have shown that $(L_*, \partial) \hookrightarrow (E_*, \partial)$ induces an isomorphism on homology. Therefore so does the inclusion $(K_*, \partial^t) \hookrightarrow (L_t^2 C_*(\Sigma_{st}), \partial^t)$. The theorem follows. \square

Let us now assume that D is a generalised homology disc. Then, along the same lines as above, one shows $L_t^2 H^*(\Sigma_{st}) \simeq L_t^2 H^*(\Sigma_{ghd})$ (as U_t -modules). More precisely, one defines $\theta: L_t^2 H^*(\Sigma_{ghd}) \rightarrow L_t^2 H^*(\Sigma_{st})$ by $\theta(\sigma) = f(\alpha)$ if $\sigma^k \subseteq \alpha^k \in \Sigma_{ghd}^{(k)}$, and $\theta(\sigma) = 0$ if no such α^k exists. The proof of $\partial^t \theta = \theta \partial^t$ is similar to that of Lemma 5.1, and it is clear that θ is a U_t -morphism. A chain $c \in L_t^2 C_k(\Sigma_{st})$ is in the image K_* of θ if and only if

- (1) c is supported by $\bigcup \Sigma_{ghd}^{(k)}$;
- (2) if $\sigma^k, \tau^k \subseteq \alpha^k \in \Sigma_{ghd}^{(k)}$, then $c(\sigma) = c(\tau)$.

These conditions do not change under m_t , and the rest of the proof of Theorem 5.3 can be repeated with dual cells replaced by cells of Σ_{ghd} (the only other change will be $[\sigma_+ : \sigma] = -[\sigma_- : \sigma]$ instead of the more complicated (5–4)). We get

Theorem 5.5 Let $(D, \partial D)$ be a generalised homology disc. Then we have the following isomorphisms of (graded) U_t -modules: $L_t^2 H^*(\Sigma_{ghd}) \simeq L_t^2 H^*(\Sigma_{st}) \simeq L_t^2 H^*(\Sigma_d)$.

6 Poincaré Duality

Let us define a map $D: L_t^2 \rightarrow L_{t^{-1}}^2$ by

$$(6-1) \quad D\left(\sum a_w \delta_w\right) = \sum (-t)^{d(w)} a_w \delta_w.$$

Direct calculation shows that D is an isometric isomorphism of Hilbert spaces. Notice that D maps $\mathbf{C}_t[W]$ onto $\mathbf{C}_{t^{-1}}[W]$. It is easy to check that D preserves the relations defining Hecke multiplication: if $d(ws) > d(w)$, then

$$D(\delta_w \delta_s) = D(\delta_{ws}) = (-t)^{d(ws)} \delta_{ws} = (-t)^{d(w)} \delta_w (-t \delta_s) = D(\delta_w) D(\delta_s);$$

if $d(ws) < d(w)$, then

$$\begin{aligned} D(\delta_w \delta_s) &= D(t \delta_{ws} + (t-1) \delta_w) = t(-t)^{d(ws)} \delta_{ws} + (t-1)(-t)^{d(w)} \delta_w \\ &= (-t)^{d(w)+1} t^{-1} \delta_{ws} + (-t)^{d(w)+1} (t^{-1} - 1) \delta_w = (-t)^{d(w)} \delta_w (-t) \delta_s \\ &= D(\delta_w) D(\delta_s). \end{aligned}$$

Hence, D restricts to an isometric isomorphism of Hilbert algebras $\mathbf{C}_t[W]$ and $\mathbf{C}_{t^{-1}}[W]$. In particular, D preserves products: for all $x, y \in \mathbf{C}_t[W]$, we have $D(xy) = D(x)D(y)$. Passing to limits with y in the norm $\|\cdot\|_t$, we deduce that the map $D: L_t^2 \rightarrow L_{t^{-1}}^2$ is a morphism of left modules over the algebra morphism $D: \mathbf{C}_t[W] \rightarrow \mathbf{C}_{t^{-1}}[W]$. Then passing to limits with x in the weak operator topology, we deduce that $D: L_t^2 \rightarrow L_{t^{-1}}^2$ is a morphism of left modules over the von Neumann algebra isomorphism $D: U_t \rightarrow U_{t^{-1}}$. Analogous statements hold for the right module structures. Finally, since D preserves the coefficient of δ_1 , it preserves dimensions of (left) submodules of L_t^2 .

Theorem 6.1 Suppose that the pair $(D, \partial D)$ is a generalised homology n -disc. Then $b_t^i = b_{t^{-1}}^{n-i}$.

Proof There is a bijection $D_T \leftrightarrow \langle T \rangle$, where $T \in \mathcal{F}$; it can be unambiguously extended to $wD_T \leftrightarrow w\langle T \rangle$, a natural bijection between i -cells of Σ_{ghd} and $(n-i)$ -cells of Σ_d . When w and T are not specified we write simply $\sigma \leftrightarrow \langle \sigma \rangle$. A property of this bijection which is crucial for us is: the codimension 1 faces of $\langle \tau^{i-1} \rangle$ are $\langle \sigma^i \rangle$, for $\sigma \supseteq \tau$. Let us pick orientations of all faces D_T of D , and extend them equivariantly to orientations of all cells η in Σ_{ghd} . Then we orient each dual cell $\langle \eta \rangle$ dually to the chosen orientation of η (dually with respect to a chosen orientation of Σ). These

orientations are of the kind considered in [Section 4](#). With these choices we have $[\langle \sigma \rangle : \langle \tau \rangle] = \pm[\sigma : \tau]$, with the sign depending only on the dimensions of σ, τ (and on n , which is fixed in our discussion).

We define the duality map $\mathcal{D}: L_t^2 C^*(\Sigma_{ghd}) \rightarrow L_{t^{-1}}^2 C^{n-*}(\Sigma_d)$ by

$$(6-2) \quad \mathcal{D}f(\langle \sigma \rangle) = t^{d(\sigma)} f(\sigma).$$

The map \mathcal{D} is an isometry of Hilbert spaces. We will now check that $\delta^{n-i}\mathcal{D} = \pm \mathcal{D}\partial_i^t$ (the sign depending only on i, n):

$$\delta(\mathcal{D}f)(\langle \tau^{i-1} \rangle) = \sum_{\sigma^i \supset \tau^{i-1}} [\langle \sigma \rangle : \langle \tau \rangle] (\mathcal{D}f)(\langle \sigma \rangle) = \pm \sum_{\sigma^i \supset \tau^{i-1}} [\sigma : \tau] t^{d(\sigma)} f(\sigma)$$

while

$$\mathcal{D}(\partial^t f)(\langle \tau^{i-1} \rangle) = t^{d(\tau)} (\partial^t f)(\tau^{i-1}) = t^{d(\tau)} \sum_{\sigma^i \supset \tau^{i-1}} [\sigma : \tau] t^{d(\sigma) - d(\tau)} f(\sigma)$$

which proves what we wanted. It follows that \mathcal{D} intertwines also the adjoint operators; consequently, it restricts to an isomorphism $\mathcal{D}: L_t^2 \mathcal{H}^*(\Sigma_{ghd}) \rightarrow L_{t^{-1}}^2 \mathcal{H}^{n-*}(\Sigma_d)$.

We still have to check that the Hecke dimensions of these spaces are the same.

To this end, let us now consider $L_t^2 C^*(\Sigma_{ghd})$ as a subspace of $\bigoplus_{T \in \mathcal{F}} L_t^2$ via the embedding Φ_t (see [Section 3](#)), and $L_{t^{-1}}^2 C^{n-*}(\Sigma_d)$ as a subspace of $\bigoplus_{T \in \mathcal{F}} L_{t^{-1}}^2$ via the embedding $\Psi_{t^{-1}}$ (see [Section 4](#)). We will check that \mathcal{D} can be regarded as the restriction of the map D (applied componentwise in $\bigoplus_{T \in \mathcal{F}} L_t^2$); it will follow that \mathcal{D} preserves dimensions. Let $f \in L^2(WD_T, \mu_t)$ be a part of a cochain on Σ_{ghd} . Then

$$\phi_T(f) = \sqrt{W_T(t)} \sum_{w \in W^T} f(wD_T) \delta_w p_T(t),$$

where $p_T(t) = \frac{1}{W_T(t)} \sum_{u \in W_T} \delta_u$. Since

$$\begin{aligned} D(p_T(t)) &= \frac{1}{W_T(t)} \sum_{u \in W_T} (-t)^{d(u)} \delta_u \\ &= \frac{1}{W_T((t^{-1})^{-1})} \sum_{u \in W_T} (-t^{-1})^{-d(u)} \delta_u = h_T(t^{-1}), \end{aligned}$$

we have

$$(6-3) \quad D(\phi_T(f)) = \sum_{w \in W^T} f(wD_T) \sqrt{W_T(t)} (-t)^{d(w)} \delta_w h_T(t^{-1}).$$

On the other hand, $(\mathcal{D}f)(w\langle T \rangle) = t^{d(w\langle T \rangle)} f(wD_T)$, and

$$(6-4) \quad \psi_T(\mathcal{D}f) = \sum_{w \in W^T} t^{d(w\langle T \rangle)} f(wD_T) (-1)^{d(w)} \sqrt{W_T(t)} \delta_w h_T(t^{-1}).$$

Since for $w \in W^T$ we have $d(w\langle T \rangle) = d(w)$, (6-3) and (6-4) are equal. \square

Remark The above proof shows that \mathcal{D} is an isomorphism of the U_t -module $L_t^2 \mathcal{H}^*(\Sigma_{ghd})$ and the $U_{t^{-1}}$ -module $L_{t^{-1}}^2 \mathcal{H}^{n-*}(\Sigma_d)$, over the algebra isomorphism $D: U_t \rightarrow U_{t^{-1}}$.

7 Calculation of b_t^0

Theorem 7.1 For $t < \rho_W$ we have $b_t^0 = \frac{1}{W(t)}$; for $t \geq \rho_W$ we have $b_t^0 = 0$.

Proof We will use the cell structure Σ_d . Vertices of Σ_d are located at the centres of chambers wD , thus they are in bijection with W . We embed $L_t^2 C^0(\Sigma_d)$ into L_t^2 by $(\Psi c)(w) = (-1)^{d(w)} c(w\langle \emptyset \rangle)$. This embedding maps all harmonic 0-cochains to constant functions, multiples of $\mathbf{1}(w) = 1$. The square of the norm of $\mathbf{1}$ is $\sum_{w \in W} t^{d(w)}$. It is finite and equal to $W(t)$ for $t < \rho_W$, and infinite if $t \geq \rho_W$. The latter means that for $t \geq \rho_W$ we have $L_t^2 \mathcal{H}^0(\Sigma_d) = 0$.

To find b_t^0 for $t < \rho_W$ we need to identify the projection of δ_1 on $L_t^2 \mathcal{H}^0(\Sigma_d)$; it is $C\mathbf{1}$, where

$$\langle \delta_1 - C\mathbf{1}, \mathbf{1} \rangle_t = 0.$$

This gives $C = \|\mathbf{1}\|_t^{-2} = \frac{1}{W(t)}$. In accordance with the procedure described at the end of [Section 2](#), we find $b_t^0 = C = \frac{1}{W(t)}$. \square

In view of [Corollary 3.4](#), the above result makes it plausible to suspect that for $t < \rho_W$ we have $b_t^{>0} = 0$. In the next section we prove that this is true for right angled Coxeter groups.

8 Mayer–Vietoris sequence

In this section we limit our attention to right angled Coxeter groups. “Right angled” means that whenever two generators $s, s' \in S$ are related in the standard presentation, they in fact commute. If we join each pair of commuting generators by an edge, we get a graph with the set of vertices S . It is convenient to fill it, gluing in a simplex whenever we can see its 1-skeleton in the graph. The resulting simplicial complex is denoted L , and the Coxeter group W_L . The Davis chamber D can be identified with the cone CL' over the first barycentric subdivision of L . We say that a subcomplex $K \subseteq L$ is full, if whenever it contains all vertices of a simplex of L , it contains the simplex as well. Full

subcomplexes K correspond to subsets of S and thus to special subgroups W_K of W_L . The Davis complex of W_K is naturally embedded in Σ_{W_L} : we first embed $D_K = CK'$ in $D_L = CL'$, and then extend W_K -equivariantly. We abbreviate Σ_{W_L} to Σ_L .

Let $L = L_1 \cup L_2$, where L_1, L_2 and (consequently) $L_0 = L_1 \cap L_2$ are full subcomplexes of L . We embed W_{L_i} into W_L , and Σ_{L_i} into Σ_L ; then $\Sigma_L = W_L \Sigma_{L_1} \cup W_L \Sigma_{L_2}$, $W_L \Sigma_{L_1} \cap W_L \Sigma_{L_2} = W_L \Sigma_{L_0}$. We have a short exact sequence of cochain complexes

$$0 \rightarrow L_t^2 C^*(\Sigma_L) \rightarrow L_t^2 C^*(W_L \Sigma_{L_1}) \oplus L_t^2 C^*(W_L \Sigma_{L_2}) \rightarrow L_t^2 C^*(W_L \Sigma_{L_0}) \rightarrow 0,$$

from which we get the long Mayer–Vietoris sequence:

$$(8-1) \quad \dots \rightarrow L_t^2 H^{i-1}(W_L \Sigma_{L_0}) \rightarrow L_t^2 H^i(\Sigma_L) \rightarrow \\ \rightarrow L_t^2 H^i(W_L \Sigma_{L_1}) \oplus L_t^2 H^i(W_L \Sigma_{L_2}) \rightarrow L_t^2 H^i(W_L \Sigma_{L_0}) \rightarrow \dots$$

Since we work with reduced cohomology, this sequence is only weakly exact (the kernels are closures of the images), see Lück [9, 1.22]. Still, if a term is preceded and followed by zero terms it has to be zero. Notice that $W_L \Sigma_{L_i}$ is the disjoint union of $w \Sigma_{L_i}$, where w runs through a set of representatives of W_{L_i} -cosets in W_L . The L_t^2 norm on $w \Sigma_{L_i}$ is $t^{d/2}$ times the L_t^2 norm on Σ_{L_i} , where d is the length of the shortest element of $w W_{L_i}$. In particular, if $L_t^2 H^p(\Sigma_{L_i}) = 0$, then $L_t^2 H^p(W_L \Sigma_{L_i}) = 0$.

Corollary 8.1 *Suppose that $b_t^{>0}(\Sigma_{L_i}) = 0$ for $i = 0, 1, 2$. Then $b_t^{>1}(\Sigma_L) = 0$.*

Theorem 8.2 *Let W be a right angled Coxeter group. For $t < \rho_W$ we have $b_t^0 = \chi_t = \frac{1}{W(t)}$ and $b_t^{>0} = 0$.*

Proof Let $W = W_L$. We argue by induction on the number of vertices of L .

- (1) If L is a simplex, then $\Sigma_{L,d}$ is a cube; its L_t^2 cohomology coincides with the usual cohomology and is concentrated in dimension 0.
- (2) If L is not a simplex, we can find two vertices $a, b \in L$ not connected by an edge; we put $L_1 = \bigcup \{\sigma \mid a \notin \sigma\}$, $L_2 = \bigcup \{\sigma \mid b \notin \sigma\}$ and $L_0 = L_1 \cap L_2$. These have fewer vertices than L , and so $L_t^2 H^{>0}(\Sigma_{L_i}) = 0$ for $t < \rho(W_{L_i})$ ($i = 0, 1, 2$). Since $L_i \subset L$, we have $\rho(W_{L_i}) \geq \rho(W_L)$. Therefore we have $L_t^2 H^{>0}(\Sigma_{L_i}) = 0$ for $t < \rho(W_L)$. It follows from Corollary 8.1 that $L_t^2 H^{>1}(\Sigma_L) = 0$ (still for $t < \rho(W_L)$), while from Corollary 3.4 and Theorem 7.1 we conclude that

$$b_t^0(\Sigma_L) = \chi_t(\Sigma_L) = b_t^0(\Sigma_L) - b_t^1(\Sigma_L).$$

Thus $b_t^1(\Sigma_L) = 0$. □

Corollary 8.3 Assume that L is a generalised homology $(n-1)$ -sphere (ie, $(D, \partial D)$ is a generalised homology n -disc); then for $t < \frac{1}{\rho(W_L)}$ we have $b_t^n = 0$, while for $t > \frac{1}{\rho(W_L)}$ the L_t^2 -cohomology is concentrated in dimension n and $b_t^n = (-1)^n \chi_t = \frac{(-1)^n}{W_L(t)}$.

Proof This follows from Theorems 8.2 and 7.1 via Poincaré duality (Theorem 6.1). \square

Proposition 8.4 Let $K \subset L$ be a full subcomplex. The dimension of the $U_t(W_L)$ -module $L_t^2 H^q(W_L \Sigma_K)$ is the same as the dimension of the $U_t(W_K)$ -module $L_t^2 H^q(\Sigma_K)$ (ie, it is equal to $b_t^q(\Sigma_K)$).

Proof A harmonic q -cochain on $W_L \Sigma_K = \bigcup \{w \Sigma_K \mid w \in W_L\}$ is the same thing as a collection of harmonic q -cochains on $w \Sigma_K$. In order to calculate dimensions, we embed everything in $V = \bigoplus_{\sigma \subset D_L} L_t^2(W_L)$. Let $\mathbf{1}_\sigma \in V$ have δ_1 as its coordinate with index σ , and 0 on all other coordinates. As we project $\mathbf{1}_{\sigma^q}$ on $L_t^2 \mathcal{H}^q(W_L \Sigma_K)$, we get in fact a harmonic cochain supported on Σ_K —harmonic cochains supported on other components of $W_L \Sigma_K$ are orthogonal to $\mathbf{1}_{\sigma^q}$, so also to its projection. We can as well project $\mathbf{1}_{\sigma^q}$ on $L_t^2 \mathcal{H}^q(\Sigma_K)$ inside $\bigoplus L_t^2(W_K)$, so that the projection matrices are the same (apart for the case $\sigma \not\subset K$, which gives 0 in the first case and does not appear in the second), and traces coincide. \square

9 Chain homotopy contraction

In this section we will describe a simplicial version of the geodesic contraction of Σ with respect to the Moussong metric. We will consider the chain complex $C_*(\Sigma_{st})$ equipped with the boundary operator ∂ given by (5–8). Henceforth we write Σ for Σ_{st} , and we denote by b the barycentre of the basic chamber D . Recall that Σ can be equipped with a W -invariant, $CAT(0)$ metric d_M , the Moussong metric (Moussong [10]). From now on, all balls, geodesics etc. will be considered with respect to d_M (unless explicitly stated otherwise). Besides $CAT(0)$, the following property of the Moussong metric will be useful for us: for every $R > 0$ there exists a constant $N(R)$ such that any ball of radius R in Σ intersects at most $N(R)$ chambers.

Theorem 9.1 There exists a linear map $H: C_*(\Sigma) \rightarrow C_{*+1}(\Sigma)$, and constants C, R , with the following properties:

- (a) if $v \in \Sigma^{(0)}$, then $\partial H(v) = v - b$;
- (b) if σ is a simplex of positive dimension, then $\partial H(\sigma) = \sigma - H(\partial\sigma)$;

- (c) for every simplex σ , $\|H(\sigma)\|_{L^\infty} < C$;
- (d) if γ is a geodesic from a vertex of a simplex σ to b , then $\text{supp}(H(\sigma)) \subseteq B_R(\text{image}(\gamma))$.

Proof We will construct, for all integers $i \geq 0$, linear maps $h_i: C_*(\Sigma) \rightarrow C_*(\Sigma)$, $H_i: C_*(\Sigma) \rightarrow C_{*+1}(\Sigma)$ such that:

- (1) $h_0 = \text{id}$;
- (2) $\partial h_i = h_i \partial$;
- (3) $\partial H_i = h_i - H_i \partial - h_{i+1}$;
- (4) $\exists C_k, \forall \sigma \in \Sigma^{(k)}, \forall i \geq 0, \|H_i(\sigma)\|_{L^\infty} < C_k$ and $\|h_i(\sigma)\|_{L^\infty} < C_k$;
- (5) $\exists R_k, \forall \sigma \in \Sigma^{(k)}, \forall i \geq 0$, if γ is a geodesic from a vertex of σ to b , then $\text{supp}(h_i(\sigma))$, $\text{supp}(H_{i-1}(\sigma))$ (if $i > 0$) and $\text{supp}(H_i(\sigma))$ are contained in the ball $B_{R_k}(\gamma(i))$ (or in $B_{R_k}(b)$, if $i > \text{length}(\gamma)$);
- (6) if $i \geq \text{diam}(\sigma \cup \{b\})$, then $h_i(\sigma) = 0$ (unless $\dim \sigma = 0$, in which case $h_i(\sigma) = b$) and $H_i(\sigma) = 0$.

The construction will be by induction on the chain degree k . Throughout this proof, we will say that a family of chains is uniformly bounded if they have uniformly bounded support diameters and L^∞ norms. Let A be the length of the longest edge in Σ .

(1) $k = 0$

Let $v \in \Sigma^{(0)}$, let $\gamma_v: [0, l] \rightarrow \Sigma$ be a geodesic such that $\gamma_v(0) = v$, $\gamma_v(l) = b$. We put $h_0(v) = v$, $h_i(v) = b$ if $i \geq l$, and we choose a vertex within distance A from $\gamma_v(i)$ and declare it to be $h_i(v)$ in the remaining cases. We have $d(h_i(v), h_{i+1}(v)) \leq 1 + 2A$. Now, up to the action of W , there are only finitely many pairs of vertices (y, z) satisfying $d(y, z) < 1 + 2A$. In every W -orbit of such pairs we choose a pair (y, z) and we fix a 1-chain $H(y, z)$, $\partial H(y, z) = y - z$; we then extend H to the W -orbit of (y, z) using the W -action (making choices if stabilisers are non-trivial). In the case $y = z$ we choose $H(y, y) = 0$. Notice that the chosen 1-chains H are uniformly bounded. Finally, we put $H_i(v) = H(h_i(v), h_{i+1}(v))$.

(2) $k \rightarrow (k + 1)$

Let $\sigma \in \Sigma^{(k+1)}$. Then, due to (2), $\partial h_i(\partial\sigma) = h_i(\partial\partial\sigma) = 0$. Thus, $h_i(\partial\sigma)$ is a cycle. Moreover, we claim that as we vary σ , the cycles $h_i(\partial\sigma)$ are uniformly bounded. In fact, as a consequence of (5), every simplex in the support of $h_i(\partial\sigma)$ is within R_k of one of the points $\gamma_v(i)$, where v runs through the vertices of σ , and, by $CAT(0)$ comparison, the $k + 2$ points $\gamma_v(i)$ are within $2A$ of each other. Whence uniform boundedness of supports of $h_i(\partial\sigma)$. Uniform boundedness of L^∞ norms follows from (4). Up to the W -action on $C_k(\Sigma)$, there are only finitely many possible values of $h_i(\partial\sigma)$. As in step 1, we fix $(k + 1)$ -chains $h_i(\sigma)$, $\partial h_i(\sigma) = h_i(\partial\sigma)$, so that they are uniformly bounded (and are 0 whenever $h_i(\partial\sigma) = 0$).

To define $H_i(\sigma)$, we consider the chain $h_i(\sigma) - H_i(\partial\sigma) - h_{i+1}(\sigma)$. It is a cycle:

$$\begin{aligned} \partial(h_i(\sigma) - H_i(\partial\sigma) - h_{i+1}(\sigma)) &= \partial h_i(\sigma) - \partial H_i(\partial\sigma) - \partial h_{i+1}(\sigma) \\ &= h_i(\partial\sigma) - (h_i(\partial\sigma) - H_i(\partial\partial\sigma) - h_{i+1}(\partial\sigma)) - h_{i+1}(\partial\sigma) = 0. \end{aligned}$$

Again, all such chains (as we vary σ) are uniformly bounded, and we can choose $H_i(\sigma)$, satisfying $\partial H_i(\sigma) = h_i(\sigma) - H_i(\partial\sigma) - h_{i+1}(\sigma)$, in a uniformly bounded way. As before, we put $H_i(\sigma) = 0$ whenever we have to chose it so that it has boundary 0 (so as to satisfy (6)).

Now that we have a family of maps satisfying (1)–(6), we put $H(\sigma) = \sum_{i \geq 0} H_i(\sigma)$. The sum is always finite because of (6). The conditions (a)–(d) are easy to check: (a) and (b) follow from (1), (3) and (6); (c) follows from (4) and (5): since the supports of $H_i(\sigma)$ are uniformly bounded and “move along” a geodesic γ with constant speed as i grows, only a uniformly finite number of $H_i(\sigma)$ contribute to a coefficient of a fixed simplex τ in the chain $H(\sigma)$; moreover, because of (4), each contribution is smaller than $C_{\dim \sigma}$; (d) is a consequence of (5). \square

10 Vanishing below ρ

Let H be a map as in [Theorem 9.1](#).

Theorem 10.1 *Suppose that $t > \frac{1}{\rho_W}$. Then the map H extends to a bounded operator $H: L_t^2 C_*(\Sigma) \rightarrow L_t^2 C_{*+1}(\Sigma)$.*

Proof Unspecified summations will be over $\Sigma^{(k)}$. N_k will denote the number of k -simplices in a chamber.

Let $a = \sum a_\sigma \sigma \in L_t^2 C_k(\Sigma)$. We know that for every simplex σ , $\|H(\sigma)\|_{L^\infty} < C$. Also

$$\begin{aligned} \sum |a_\sigma| &= \sum |a_\sigma| t^{d(\sigma)/2} t^{-d(\sigma)/2} \leq \left(\sum |a_\sigma|^2 t^{d(\sigma)} \right)^{1/2} \left(\sum t^{-d(\sigma)} \right)^{1/2} \\ &\leq \|a\|_t (N_k W(t^{-1}))^{1/2} < +\infty, \end{aligned}$$

so that $\sum a_\sigma H(\sigma)$ is pointwise convergent to a chain $H(a) \in L^\infty C_{k+1}(\Sigma)$. We want to estimate $\|H(a)\|_t$. Let us write $\tau \prec \sigma$ if τ appears with non-zero coefficient in $H(\sigma)$. We have $|H(a)_\tau| \leq \sum_{\sigma|\tau \prec \sigma} C |a_\sigma|$, so that

$$\begin{aligned} \sum |H(a)_\tau|^2 t^{d(\tau)} &\leq C^2 \sum_{\tau} \left(\sum_{\sigma|\tau \prec \sigma} |a_\sigma| \right)^2 t^{d(\tau)} \\ (10-1) \quad &\leq C^2 \sum_{\tau} \left(\sum_{\sigma|\tau \prec \sigma} |a_\sigma| t^{d(\sigma)/2} t^{-\alpha(\frac{d(\sigma)-d(\tau)}{2})} t^{-\beta(\frac{d(\sigma)-d(\tau)}{2})} \right)^2 \\ &\leq C^2 \sum_{\tau} \left(\sum_{\sigma|\tau \prec \sigma} |a_\sigma|^2 t^{d(\sigma)} (t^{-\alpha})^{d(\sigma)-d(\tau)} \right) \left(\sum_{\sigma|\tau \prec \sigma} (t^{-\beta})^{d(\sigma)-d(\tau)} \right). \end{aligned}$$

Here α, β are positive numbers chosen so that $\alpha + \beta = 1$, $t^{-\beta} < \rho_W$.

Claim *There exists a constant C' , independent of τ , such that*

$$\sum_{\sigma|\tau \prec \sigma} (t^{-\beta})^{d(\sigma)-d(\tau)} \leq C' W(t^{-\beta}).$$

Proof Recall that A is the length of the longest edge in Σ , and $N(r)$ is the maximal number of chambers intersecting a ball of radius r . The claim follows from two observations.

(1) For $w_0 \in W$ let $E(w_0) = \{w \in W \mid d(w) = d(w_0) + d(w_0^{-1}w)\}$. In more geometric terms, $E(w_0)$ is the set of all w such that some gallery connecting D and wD passes through w_0D . We have

$$\sum_{w \in E(w_0)} (t^{-\beta})^{d(w)-d(w_0)} = \sum_{w \in E(w_0)} (t^{-\beta})^{d(w_0^{-1}w)} \leq \sum_{w \in W} (t^{-\beta})^{d(w)} = W(t^{-\beta}).$$

(2) If $\tau \prec \sigma$, then τ is at distance at most R from a geodesic γ joining (a vertex of) σ and b . Let us consider the union U of all galleries joining D and a fixed chamber D' containing σ . Then U is the intersection of all half-spaces containing D and D' (see Ronan [11]). Since half-spaces are geodesically convex in d_M , we have $\gamma \subseteq U$. Consequently, every point of γ lies in a gallery joining D' and D . Therefore, if we put $B(\tau) = \{w_0 \mid w_0D \cap B_R(\tau) \neq \emptyset\}$, then we have $\{\sigma \mid \tau \prec \sigma\} \subseteq \bigcup_{w_0 \in B(\tau)} E(w_0)D$.

Putting these together,

$$\begin{aligned} \sum_{\sigma \mid \tau \prec \sigma} (t^{-\beta})^{d(\sigma)-d(\tau)} &\leq \sum_{w_0 \in B(\tau)} t^{-\beta(d(w_0)-d(\tau))} \sum_{w \in E(w_0)} N_k (t^{-\beta})^{d(w)-d(w_0)} \\ &\leq \sum_{w_0 \in B(\tau)} t^{-\beta(d(w_0)-d(\tau))} N_k W(t^{-\beta}). \end{aligned}$$

Notice that $|d(w_0) - d(\tau)|$ does not exceed the gallery distance from w_0D to some chamber containing τ , and is therefore uniformly bounded. Also, the cardinality of $B(\tau)$ is bounded by $N(R + A)$. The claim is proved. \square

Using the claim, we can continue the estimate (10–1):

$$\|H(a)\|_t^2 \leq C^2 C' W(t^{-\beta}) \sum_{\tau} \left(\sum_{\sigma \mid \tau \prec \sigma} |a_{\sigma}|^2 t^{d(\sigma)} (t^{-\alpha})^{d(\sigma)-d(\tau)} \right).$$

Now

$$\sum_{\tau} \left(\sum_{\sigma \mid \tau \prec \sigma} |a_{\sigma}|^2 t^{d(\sigma)} (t^{-\alpha})^{d(\sigma)-d(\tau)} \right) = \sum_{\sigma} \left(|a_{\sigma}|^2 t^{d(\sigma)} \sum_{\tau \mid \tau \prec \sigma} (t^{-\alpha})^{d(\sigma)-d(\tau)} \right),$$

so that the following lemma is all we need:

Lemma 10.2 *There exists a constant K independent of σ such that*

$$\sum_{\tau \mid \tau \prec \sigma} (t^{-\alpha})^{d(\sigma)-d(\tau)} < K.$$

Proof Since W acts on (Σ, d_M) isometrically, cocompactly and properly discontinuously, the word metric d on W is quasi-isometric to the metric d_M restricted to $W \simeq Wb \hookrightarrow \Sigma$. This implies that there exist constants M, m, L such that for any two points $y, z \in \Sigma$ and any chambers $D_y \ni y, D_z \ni z$, we have

$$(10-2) \quad M d_M(y, z) + L \geq d(D_y, D_z) \geq m d_M(y, z) - L,$$

where we put $d(wD, uD) = d(w, u) = d(w^{-1}u)$.

Let v be a vertex of σ , and let $\gamma: [0, l] \rightarrow \Sigma$ be a geodesic, $\gamma(0) = v, \gamma(l) = b$. To each $\tau \prec \sigma$ we can assign one of the points $\gamma(i)$ ($0 \leq i \leq \lfloor l \rfloor$) in such a way that $d_M(\tau, \gamma(i)) < R + 1$. The number of simplices to which we assign a given $\gamma(i)$ does not exceed $N(R + 1)N_k$. Suppose that $\gamma(i)$ is assigned to τ . Let D_τ (resp. D_σ) be the chamber containing τ (resp. σ) such that $d(\tau) = d(D, D_\tau)$ (resp. $d(\sigma) = d(D, D_\sigma)$). Let D_i be a chamber containing $\gamma(i)$. We choose D_i so that some gallery from D to D_σ passes through D_i (see part 2 of the proof of the claim above). Using (10-2), we get

$$\begin{aligned} d(\sigma) - d(\tau) &= d(D, D_\sigma) - d(D, D_\tau) \\ &\geq d(D, D_i) + d(D_i, D_\sigma) - (d(D, D_i) + d(D_i, D_\tau)) \\ &\geq m d_M(\gamma(i), v) - L - (M d_M(\tau, \gamma(i)) + L) \\ &\geq m i - (M(R + 1) + 2L) = m i - P, \end{aligned}$$

where $P = M(R + 1) + 2L$. Remember that t^{-1} and, whence, $t^{-\alpha}$ are less than 1. Therefore

$$\begin{aligned} \sum_{\tau \mid \tau \prec \sigma} (t^{-\alpha})^{d(\sigma) - d(\tau)} &\leq \sum_{i=0}^{\lfloor l \rfloor} N(R + 1)N_k (t^{-\alpha})^{m i - P} \\ &= N(R + 1)N_k t^{\alpha P} \sum_{i=0}^{\lfloor l \rfloor} (t^{-\alpha m})^i \\ &\leq N(R + 1)N_k t^{\alpha P} \frac{1}{1 - t^{-\alpha m}}. \end{aligned}$$

This completes the proof of Lemma 10.2 and of Theorem 10.1. \square

Theorem 10.3 *Let W be a Coxeter group. For $t < \rho_W$ we have $b_t^0(W) = \chi_t(W) = \frac{1}{W(t)}$ and $b_t^{>0}(W) = 0$.*

Proof Theorems 9.1 and 10.1 imply that in the range $t > \frac{1}{\rho_W}$ we have

$$H_{>0}(L_t^2 C_*(\Sigma), \partial) = 0.$$

Indeed, if $c \in L_t^2 C_k(\Sigma)$, $\partial c = 0$, then $c = \partial H(c) + H(\partial c) = \partial H(c)$, so that $[c] = 0$. It follows that the isomorphic complex $(L_{t-1}^2 C_*(\Sigma), \partial^{t-1})$ also has vanishing homology in degrees > 0 (if $t^{-1} < \rho_W$). Thus, its homology is concentrated in dimension 0, and the zeroth Betti number is equal to the Euler characteristic. \square

Corollary 10.4 *Assume that $(D, \partial D)$ is a generalised homology n -disc; then for $t < \frac{1}{\rho_W}$ we have $b_t^n = 0$, while for $t > \frac{1}{\rho_W}$ the L_t^2 cohomology is concentrated in dimension n and $b_t^n = (-1)^n \chi_t = \frac{(-1)^n}{W(t)}$.*

Proof This follows from Theorems 10.3 and 7.1 using Poincaré duality (Theorem 6.1). \square

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